

Bell inequality for multipartite qubit quantum system and the maximal violation

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Abstract We present a set of Bell inequalities for multi-qubit quantum systems. These Bell inequalities are shown to be able to detect multi-qubit entanglement better than previous Bell inequalities such as WWZB ones. Computable formulas are presented for calculating the maximal violations of these Bell inequalities for any multi-qubit states.

Keywords Bell inequality; Entanglement; Separability.

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Based on the Einstein, Podolsky and Rosen (EPR) gedanken experiment, Bell presented an inequality obeyed by any local hidden variable theory [1]. It turns out that this inequality and its generalized forms are satisfied by all separable quantum states, but may be violated by pure entangled states and some mixed quantum states [2, 3, 4, 5, 6, 7]. Thus Bell inequalities are of great importance both in understanding the conceptual foundations of quantum theory and in investigating quantum entanglement. Bell inequalities are also closely related to certain tasks in quantum information processing, such as building quantum protocols to decrease communication complexity [8] and providing secure quantum communication [9]. Due to their significance, Bell inequalities have been generalized from two qubit case, such as the Clauser-Horne-Shimony-Holt (CHSH) inequality [10] to the N -qubit case, such as the Mermin-Ardehali-Belinskii-Klyshko (MABK) inequality [11, 12], and to arbitrary d -dimensional (qudit) systems such as the Collins-Gisin-Linden-Masser-Popescu (CGLMP) inequality [13]. However, except for some special cases such as bipartite pure states [2, 3, 6], three-qubit pure states [5, 14], and general two-qubit quantum states [15], there are no Bell inequalities yet that can be violated by all the entangled quantum states, although it is shown recently that any entangled multipartite pure states should violate a Bell inequality [7]. Thus it is of great importance to find more effective Bell type inequalities to detect the quantum entanglement.

In this paper, we study Bell inequalities for both pure and mixed multi-qubit systems. We propose a series of Bell inequalities for any N -qubit states ($N \geq 3$), and derive the formulas of the maximal violations of these Bell inequalities. This gives a sufficient and necessary condition which is also practical for any multipartite qubits quantum states. It is shown that the Bell inequalities constructed in this paper are independent of the WWZB inequality and Chen's Bell inequalities

constructed in (3), i.e. they can detect some entangled states which fulfill both the WWZB inequality and Chen's Bell inequalities.

Consider an N -qubit quantum system and allow each part to choose independently between two dichotomic observables A_i, A'_i for the i th observer, $i = 1, 2, \dots, N$. Each measurement has two possible outcomes 1 and -1 . Quantum mechanically these observables can be expressed as $A_i = \vec{a}_i \vec{\sigma}, A'_i = \vec{a}'_i \vec{\sigma}$, where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices and \vec{a}_i, \vec{a}'_i are unit vectors, $i = 1, 2, \dots, N$.

The CHSH Bell inequality for two-qubit systems is given by

$$|\langle B_2 \rangle| \leq 1, \quad (1)$$

where the Bell operator $B_2 = \frac{1}{2}[A_1 A_2 + A'_1 A_2 + A_1 A'_2 - A'_1 A'_2]$. In [16] Horodeckis have derived an elegant formula which serves as a necessary and sufficient condition for violating the CHSH inequality by an arbitrary mixed two qubits state.

The WWZB Bell operator is defined by

$$B_N^{WWZB} = \frac{1}{2^N} \sum_{s_1, s_2, \dots, s_N = \pm 1} S(s_1, s_2, \dots, s_N) \sum_{k_1, k_2, \dots, k_N = \pm 1} s_1^{k_1} s_2^{k_2} \dots s_N^{k_N} \otimes_{j=1}^N O_j(k_j), \quad (2)$$

where $S(s_1, s_2, \dots, s_N)$ is an arbitrary function of $s_i (= \pm 1)$, $i = 1, \dots, N$, taking values ± 1 , $O_j(1) = A_j$ and $O_j(2) = A'_j$ with $k_j = 1, 2$. It is shown in [17, 18] that local realism requires $|\langle B_N \rangle| \leq 1$. The MABK inequality is recovered by taking $S(s_1, s_2, \dots, s_N) = \sqrt{2} \cos[(s_1 + s_2 + \dots + s_N - N + 1)/\frac{\pi}{4}]$ in (2). In [17, 18] the authors also derived a necessary and sufficient condition of violation of this inequality for an arbitrary N -qubit mixed state, generalizing two-qubit results in [16]. However, when using the results to obtain the maximal violation of the WWZB inequality, one has to select a proper set of local coordinate systems and a proper set of unit vectors, which makes the approach less operational.

Employing an inductive method from the $(N-1)$ -partite WWZB Bell inequality to the N -partite inequality, a family of Bell inequalities was presented in [12]. The Bell operator is defined by

$$B_N = B_{N-1}^{WWZB} \otimes \frac{1}{2}(A_N + A'_N) + I_{N-1} \otimes \frac{1}{2}(A_N - A'_N), \quad (3)$$

where B_{N-1}^{WWZB} represents the normal WWZB Bell operators defined in (2), I_{N-1} be the identity operators acting on first $(N-1)$ qubits. Such Bell operators yield the violation of the Bell inequality for the generalized GHZ states, $|\psi\rangle = \cos \alpha |00 \dots 0\rangle + \sin \alpha |11 \dots 1\rangle$, in the whole parameter region of α such that $\cos \alpha \neq 0$ and $\sin \alpha \neq 0$, and for any number of qubits, thus overcoming the drawback of the WWZB inequality. In the three-qubit case, one can construct three different Bell operators from B_2 by using the approach of (3). The corresponding three Bell inequalities can distinguish full separability, detailed partial separability and true entanglement [19]. However, the maximal violation of this Bell inequality is unknown for a generally given three-qubit state.

We start with constructing a set of new Bell inequalities for any N -qubit quantum systems by iteration. First consider the case $N = 3$. As a two-qubit CHSH Bell operator \mathcal{B}_2 can act on two of

the three qubits in three different ways, we can have three Bell operators,

$$\mathcal{B}_3^i = (\mathcal{B}_2)^i \otimes \frac{1}{2}(A_i + A'_i) + (I_2)^i \otimes \frac{1}{2}(A_i - A'_i), \quad i = 1, 2, 3, \quad (4)$$

where $(\mathcal{B}_2)^i$ and $(I_2)^i$ are the two-qubit CHSH Bell operator and the identity operator acting on the two qubits except for the i th one. For $N \geq 4$, the Bell operators can be similarly obtained,

$$\mathcal{B}_N^{(i-1)\frac{(N-1)!}{2}+j} = (\mathcal{B}_{N-1}^j)^i \otimes \frac{1}{2}(A_i + A'_i) + (I_{N-1})^i \otimes \frac{1}{2}(A_i - A'_i), \quad (5)$$

with $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, \frac{(N-1)!}{2}$. Here $(\mathcal{B}_{N-1}^j)^i$ denotes the j th Bell operator acting on the $(N-1)$ qubits except for the i th one. $(I_{N-1})^i$ stands for the identity operator acting on the $(N-1)$ qubits except for the i th one. There are totally $\frac{N!}{2}$ Bell operators.

Theorem 1: If a local realistic description is assumed, the following inequalities must hold,

$$|\langle \mathcal{B}_N^k \rangle| \leq 1, \quad (6)$$

where $k \in \{1, 2, \dots, \frac{N!}{2}\}$.

Proof: We prove the theorem by induction. Note that for two qubits systems, local realism requires that $|\langle \mathcal{B}_2 \rangle| \leq 1$ as shown in (1). Assume that a local realistic model has lead to $|\langle \mathcal{B}_{N-1}^k \rangle| \leq 1$ with $k \in \{1, 2, \dots, \frac{(N-1)!}{2}\}$. We consider the N -partite systems. If A_i and A'_i are specified by some local parameters each having two possible outcomes -1 and 1 , one has either $|A_i + A'_i| = 2$ and $|A_i - A'_i| = 0$, or vice versa. For any $k \in \{1, 2, \dots, \frac{N!}{2}\}$, from (5) we have that

$$\begin{aligned} |\langle \mathcal{B}_N^k \rangle| &= |\langle \mathcal{B}_N^{(i-1)\frac{(N-1)!}{2}+j} \rangle| = |\langle (\mathcal{B}_{N-1}^j)^i \otimes \frac{1}{2}(A_i + A'_i) + (I_{N-1})^i \otimes \frac{1}{2}(A_i - A'_i) \rangle| \\ &\leq |\langle (\mathcal{B}_{N-1}^j)^i \rangle| \otimes \frac{1}{2}|\langle (A_i + A'_i) \rangle| + \frac{1}{2}|\langle (A_i - A'_i) \rangle| \leq 1. \end{aligned}$$

□

It is shown in [20] that violation of a Bell inequality gives rise formally to a kind of entanglement witness. Moreover, the separability criterion and the existence of a description of the state by a local hidden variable theory will become equivalent when one restricts the set of local hidden variable theories to the domain of quantum mechanics. Thus one can use the Bell inequalities as the separability criteria to detect quantum entanglement. We remark that any N -qubit fully separable states also satisfy the inequality (6). For $N \geq 4$, the operator \mathcal{B}_{N-1}^i is derived from \mathcal{B}_{N-2}^i . Thus \mathcal{B}_N^i are different from the Bell operators in [12] where \mathcal{B}_{N-1}^i is the Bell operator in the WWZB inequality. The following example will show that our Bell inequalities in (6) are independent from the WWZB inequalities and that in [12], and our new Bell inequalities can detect entanglement better than they can.

Example Consider a four-qubit pure state $|\psi\rangle = |\phi\rangle \otimes |0\rangle$, where $|\phi\rangle = \cos \alpha |000\rangle + \sin \alpha |111\rangle$, $\alpha \in [0, \frac{\pi}{12}]$. It has been proved [21] that for $\sin 2\alpha \leq \frac{1}{2}$ (i.e. $\alpha \in [0, \frac{\pi}{12}]$), the WWZB Bell inequalities cannot be violated by the generalized GHZ state $|\phi\rangle$. According to the result in [21], the WWZB inequalities operator B_4^{WWZB} and the Bell operator B_4 in [12] satisfy the following relations,

$$|\langle \psi | B_4^{WWZB} | \psi \rangle| \leq |\langle \phi | B_3^{WWZB} | \phi \rangle| \leq 1, \quad (7)$$

$$|\langle\psi|B_4|\psi\rangle| \leq |\langle\phi|B_3^{WWZB}|\phi\rangle| \leq 1. \quad (8)$$

Therefore both the WWZB Bell inequalities and the inequalities in [12] can not detect entanglement of $|\psi\rangle$.

Nevertheless the mean values of the Bell operator \mathcal{B}_4^{12} in (6) is $\sqrt{2\sin^2 2\alpha + \cos^2 2\alpha}$ which is always larger than 1 as long as $|\phi\rangle$ is not separable. Therefore the entanglement is detected by our Bell inequality (6).

To identify the non-local properties of a quantum state, it is important to find the necessary and sufficient conditions for the quantum state to violate the Bell inequality. Now we investigate the maximal violation of the Bell inequalities (6). We first consider the $N = 3$ case. In this situation, (5) gives three operators,

$$\mathcal{B}_3^1 = (\mathcal{B}_2)^1 \otimes \frac{1}{2}(A_1 + A'_1) + (I_2)^1 \otimes \frac{1}{2}(A_1 - A'_1), \quad (9)$$

$$\mathcal{B}_3^2 = (\mathcal{B}_2)^2 \otimes \frac{1}{2}(A_2 + A'_2) + (I_2)^2 \otimes \frac{1}{2}(A_2 - A'_2), \quad (10)$$

$$\mathcal{B}_3^3 = (\mathcal{B}_2)^3 \otimes \frac{1}{2}(A_3 + A'_3) + (I_2)^3 \otimes \frac{1}{2}(A_3 - A'_3), \quad (11)$$

where $(\mathcal{B}_2)^1 = \frac{1}{2}(A_2A_3 + A'_2A_3 + A_2A'_3 - A'_2A'_3)$, $(\mathcal{B}_2)^2 = \frac{1}{2}(A_1A_3 + A'_1A_3 + A_1A'_3 - A'_1A'_3)$ and $(\mathcal{B}_2)^3 = \frac{1}{2}(A_1A_2 + A'_1A_2 + A_1A'_2 - A'_1A'_2)$. Let ρ be a general three-qubit state,

$$\rho = \sum_{i,j,k=0}^3 T_{ijk} \sigma_i \sigma_j \sigma_k, \quad (12)$$

where $\sigma_0 = I_2$ is the 2×2 identity matrix, σ_i are the Pauli matrices, and

$$T_{ijk} = \frac{1}{8} \text{Tr}(\rho \sigma_i \sigma_j \sigma_k). \quad (13)$$

Theorem 2: The maximum of the mean values of the Bell operators in (9), (10) and (11) satisfy the following relations,

$$\max |\langle \mathcal{B}_3^1 \rangle| = 8 \max \{ \lambda_1^1(\vec{b}_3) + \lambda_2^1(\vec{b}_3) + \|\vec{T}_{00}^1\|^2 - \langle \vec{b}_3, \vec{T}_{00}^1 \rangle^2 \}^{\frac{1}{2}}, \quad (14)$$

$$\max |\langle \mathcal{B}_3^2 \rangle| = 8 \max \{ \lambda_1^2(\vec{b}_3) + \lambda_2^2(\vec{b}_3) + \|\vec{T}_{00}^2\|^2 - \langle \vec{b}_3, \vec{T}_{00}^2 \rangle^2 \}^{\frac{1}{2}}, \quad (15)$$

$$\max |\langle \mathcal{B}_3^3 \rangle| = 8 \max \{ \lambda_1^3(\vec{b}_3) + \lambda_2^3(\vec{b}_3) + \|\vec{T}_{00}^3\|^2 - \langle \vec{b}_3, \vec{T}_{00}^3 \rangle^2 \}^{\frac{1}{2}}, \quad (16)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors, $\|\vec{x}\|$ stands for the norm of vector \vec{x} . The maximums on the right of (14), (15) and (16) are taken over all the unit vectors \vec{b}_3 . Given a three-qubit state ρ , one can compute T_{ijk} by using the formula in (13). Then $\lambda_1^i(\vec{b}_3)$ and $\lambda_2^i(\vec{b}_3)$ are defined to be the two greater eigenvalues of the matrix $M_i^\dagger M_i$ with $M_i = \sum_{k=1}^3 b_3^k T_k^i$, $i = 1, 2, 3$, with respect to the three Bell operators in (9), (10) and (11). Here T_k^l , $l = 1, 2, 3$, are matrices with entries given

by $(T_k^1)_{ij} = T_{kij}$, $(T_k^2)_{ij} = T_{ikj}$ and $(T_k^3)_{ij} = T_{ijk}$. \vec{T}_{00}^m , $m = 1, 2, 3$ are defined to be vectors with entries $(\vec{T}_{00}^1)_k = T_{k00}$, $(\vec{T}_{00}^2)_k = T_{0k0}$ and $(\vec{T}_{00}^3)_k = T_{00k}$.

Proof: We take (11) as an example to show how to calculate the maximal violation. The maximal violation for the Bell operators (9) and (10) can be computed similarly. A direct computation shows that

$$\begin{aligned}\mathcal{B}_3^3 &= \frac{1}{4}[(A_1 + A'_1)A_2 + (A_1 - A'_1)A'_2](A_3 + A'_3) + (I_2)^3 \otimes \frac{1}{2}(A_3 - A'_3) \\ &= \frac{1}{4}(A_1 + A'_1)A_2(A_3 + A'_3) + \frac{1}{4}(A_1 - A'_1)A'_2(A_3 + A'_3) + (I_2)^3 \otimes \frac{1}{2}(A_3 - A'_3) \\ &= \frac{1}{4} \sum_{i,j,k=1}^3 [a_1^i + (a'_1)^i]a_2^j[a_3^k + (a'_3)^k]\sigma_i\sigma_j\sigma_k + \frac{1}{4} \sum_{i,j,k=1}^3 [a_1^i - (a'_1)^i](a'_2)^j[a_3^k + (a'_3)^k]\sigma_i\sigma_j\sigma_k \\ &\quad + \frac{1}{2} \sum_{k=1}^3 [a_3^k - (a'_3)^k]I_4 \otimes \sigma_k.\end{aligned}\tag{17}$$

For any given unit vectors \vec{a}_1 and \vec{a}'_1 , there always exist a pair of unit and mutually orthogonal vectors \vec{b}_1, \vec{b}'_1 and $\theta \in [0, \frac{\pi}{2}]$ such that

$$\vec{a}_1 + \vec{a}'_1 = 2 \cos \theta \vec{b}_1, \quad \vec{a}_1 - \vec{a}'_1 = 2 \sin \theta \vec{b}'_1.\tag{18}$$

Similarly for \vec{a}_3 and \vec{a}'_3 , we have

$$\vec{a}_3 + \vec{a}'_3 = 2 \cos \phi \vec{b}_3, \quad \vec{a}_3 - \vec{a}'_3 = 2 \sin \phi \vec{b}'_3,\tag{19}$$

where \vec{b}_3 and \vec{b}'_3 are orthogonal vectors with unit norm and $\phi \in [0, \frac{\pi}{2}]$.

By inserting (12) into (22) we get the mean value of the Bell operator (11),

$$\begin{aligned}\langle \mathcal{B}_3^3 \rangle &= \text{Tr}(\rho \mathcal{B}_3^3) \\ &= \frac{1}{4} \sum_{i,j,k=1}^3 [a_1^i + (a'_1)^i]a_2^j[a_3^k + (a'_3)^k]T_{ijk}\text{Tr}(\sigma_i^2\sigma_j^2\sigma_k^2) \\ &\quad + \frac{1}{4} \sum_{i,j,k=1}^3 [a_1^i - (a'_1)^i](a'_2)^j[a_3^k + (a'_3)^k]T_{ijk}\text{Tr}(\sigma_i^2\sigma_j^2\sigma_k^2) \\ &\quad + \frac{1}{2} \sum_{k=1}^3 [a_3^k - (a'_3)^k]T_{00k}\text{Tr}(I_4^2 \otimes \sigma_k^2) \\ &= 8 \sum_{i,j,k=1}^3 b_1^i b_3^k a_2^j T_{ijk} \cos \theta \cos \phi + 8 \sum_{i,j,k=1}^3 (b'_1)^i b_3^k (a'_2)^j T_{ijk} \sin \theta \cos \phi + 4 \sum_{k=1}^3 (b'_3)^k T_{00k} \sin \phi.\end{aligned}$$

Let $T_k^3, k = 1, 2, 3$, be the matrix with entries given by $(T_k^3)_{ij} = T_{ijk}$ and \vec{T}_{00}^3 a vector with components $(\vec{T}_{00}^3)_k = T_{00k}$. The maximal mean value of the Bell operator (11) can be written as

$$\max \langle \mathcal{B}_3^3 \rangle = 8 \max[\langle \vec{b}_1, \sum_{k=1}^3 b_3^k T_k^3 \vec{a}_2 \rangle \cos \theta \cos \phi + \langle \vec{b}'_1, \sum_{k=1}^3 b_3^k T_k^3 \vec{a}'_2 \rangle \sin \theta \cos \phi + \langle \vec{b}_3, \vec{T}_{00}^3 \rangle \sin \phi]$$

$$\begin{aligned}
&= 8 \max\{[\langle \vec{b}_1, \sum_{k=1}^3 b_3^k T_k^3 \vec{d}_2 \rangle \cos \theta + \langle \vec{b}_1, \sum_{k=1}^3 b_3^k T_k^3 \vec{d}_2' \rangle \sin \theta]^2 + \langle \vec{b}_3, \vec{T}_{00}^3 \rangle^2\}^{\frac{1}{2}} \\
&= 8 \max\{[\langle \vec{b}_1, \sum_{k=1}^3 b_3^k T_k^3 \vec{d}_2 \rangle \cos \theta + \langle \vec{b}_1, \sum_{k=1}^3 b_3^k T_k^3 \vec{d}_2' \rangle \sin \theta]^2 + \|\vec{T}_{00}^3\|^2 - \langle \vec{b}_3, \vec{T}_{00}^3 \rangle^2\}^{\frac{1}{2}} \\
&= 8 \max\{\lambda_1^3(\vec{b}_3) + \lambda_2^3(\vec{b}_3) + \|\vec{T}_{00}^3\|^2 - \langle \vec{b}_3, \vec{T}_{00}^3 \rangle^2\}^{\frac{1}{2}}, \tag{20}
\end{aligned}$$

which proves (16). In (20) we have used the fact that the maximum of $x \cos \theta + y \sin \theta$ taking over all θ is $\sqrt{x^2 + y^2}$. Formulae (14) and (15) can be similarly proven. \square

Remark: According to the symmetry of the operator \mathcal{B}_3^3 , the equation (20) also provides the minimum of the operator (11), achieved by $-\mathcal{B}_3^3$.

Since \vec{b}_3 is a three dimensional real unit vector, one can always calculate the exact value of the maximum for any given three qubits quantum state. For example, for the generalized three-qubit GHZ state, $|GHZ\rangle = \cos \alpha |000\rangle + \sin \alpha |111\rangle$, by selecting some proper direction of the measurement operators, i.e. \vec{d}_i s and $(\vec{d}')_i$ s, the maximal mean value of the Bell operator in (11) is shown to be [12], $\sqrt{2 \sin^2 2\alpha + \cos^2 2\alpha}$. From our formulae in Theorem 2 one can show that the result are in accord with that in [12]. For three-qubit W state, $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, our mean value is 1.202, which is also in agreement with that in [12]. However, our method can be also used to calculate the mean value of the Bell operators in (9), (10) and (11) for any three qubits quantum states. For instance, we consider the mixture of $|W\rangle$ and $|GHZ\rangle$,

$$\rho = x|W\rangle\langle W| + (1-x)|GHZ\rangle\langle GHZ|, \tag{21}$$

where $0 \leq x \leq 1$. We have the maximal mean value of the Bell operator in (11), see Fig.1, where $f(x)$ stands for the maximal mean value of the operator (11) for the mixed state ρ . For $0 \leq x \leq 0.33$ and $0.82 \leq x \leq 1$, $f(x) > 1$ and ρ is detected to be entangled.

For four-qubit systems, the Bell operators (5) have four different forms. We take \mathcal{B}_4^{12} as an example to investigate the maximal violation of the corresponding Bell inequality. Note that

$$\begin{aligned}
\mathcal{B}_4^{12} &= (\mathcal{B}_3^3)^4 \otimes \frac{1}{2}(A_4 + A_4') + (I_3)^4 \otimes \frac{1}{2}(A_4 - A_4') \\
&= \frac{1}{8}(A_1 + A_1')A_2(A_3 + A_3')(A_4 + A_4') + \frac{1}{8}(A_1 - A_1')A_2'(A_3 + A_3')(A_4 + A_4') \\
&\quad + \frac{1}{4}I_4 \otimes (A_3 - A_3')(A_4 + A_4') + \frac{1}{2}I_6 \otimes (A_4 - A_4') \\
&= \frac{1}{8} \sum_{i,j,k,l=1}^3 [a_1^i + (a_1')^i]a_2^j[a_3^k + (a_3')^k][a_4^l + (a_4')^l]\sigma_i\sigma_j\sigma_k\sigma_l \\
&\quad + \frac{1}{8} \sum_{i,j,k,l=1}^3 [a_1^i - (a_1')^i](a_2')^j[a_3^k + (a_3')^k][a_4^l + (a_4')^l]\sigma_i\sigma_j\sigma_k\sigma_l \\
&\quad + \frac{1}{4} \sum_{k,l=1}^3 [a_3^k - (a_3')^k][a_4^l + (a_4')^l]I_4 \otimes \sigma_k\sigma_l + \frac{1}{2} \sum_{l=1}^3 [a_4^l - (a_4')^l]I_6 \otimes \sigma_l. \tag{22}
\end{aligned}$$

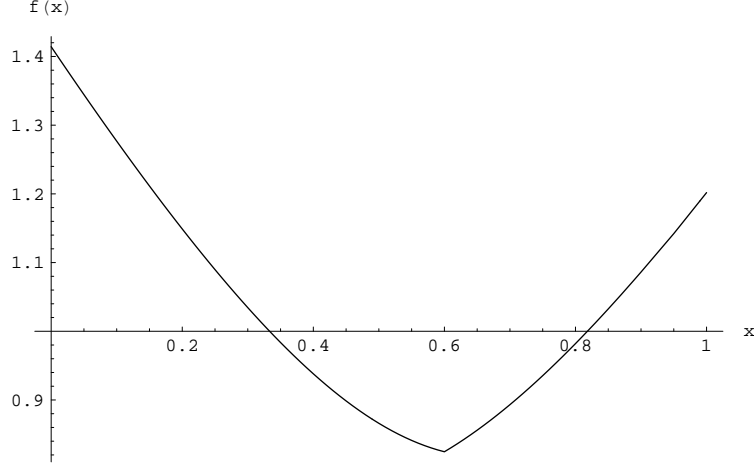


Figure 1: The maximal mean value of the operator (11) for the mixed state ρ in (21). $f(x)$ stands for the maximal mean value and x is the parameter in ρ .

Let ρ be a general four-qubit quantum state,

$$\rho = \sum_{i,j,k,l=0}^3 T_{ijkl} \sigma_i \sigma_j \sigma_k \sigma_l, \quad (23)$$

with $T_{ijkl} = \frac{1}{2^4} \text{Tr}(\rho \sigma_i \sigma_j \sigma_k \sigma_l)$. The mean value of \mathcal{B}_4^{12} can be derived by the following deduction.

$$\begin{aligned} \langle \mathcal{B}_4^{12} \rangle &= \frac{1}{8} \sum_{i,j,k,l=1}^3 [a_1^i + (a'_1)^i] a_2^j [a_3^k + (a'_3)^k] [a_4^l + (a'_4)^l] T_{ijkl} \text{Tr}(\sigma_i^2 \sigma_j^2 \sigma_k^2 \sigma_l^2) \\ &\quad + \frac{1}{8} \sum_{i,j,k,l=1}^3 [a_1^i - (a'_1)^i] (a'_2)^j [a_3^k + (a'_3)^k] [a_4^l + (a'_4)^l] T_{ijkl} \text{Tr}(\sigma_i^2 \sigma_j^2 \sigma_k^2 \sigma_l^2) \\ &\quad + \frac{1}{4} \sum_{k,l=1}^3 [a_3^k - (a'_3)^k] [a_4^l + (a'_4)^l] T_{00kl} \text{Tr}(I_4 \otimes \sigma_k^2 \sigma_l^2) \\ &\quad + \frac{1}{2} \sum_{l=1}^3 [a_4^l - (a'_4)^l] T_{000l} \text{Tr}(I_6 \otimes \sigma_l^2) \\ &= 2^4 \sum_{i,j,k,l=1}^3 b_1^i a_2^j b_3^k b_4^l T_{ijkl} \cos \alpha_1 \cos \alpha_3 \cos \alpha_4 \\ &\quad + 2^4 \sum_{i,j,k,l=1}^3 (b'_1)^i (a'_2)^j b_3^k b_4^l T_{ijkl} \sin \alpha_1 \cos \alpha_3 \cos \alpha_4 \\ &\quad + 2^4 \sum_{k,l=1}^3 (b'_3)^k b_4^l T_{00kl} \sin \alpha_3 \cos \alpha_4 + 2^4 \sum_{l=1}^3 (b'_4)^l T_{000l} \sin \alpha_4, \end{aligned}$$

where we have assumed that $\vec{a}_i + \vec{a}'_i = 2 \cos \alpha_i \vec{b}_i$, $\vec{a}_i - \vec{a}'_i = 2 \sin \alpha_i \vec{b}_i$, $\alpha_i \in [0, \frac{\pi}{2}]$.

The maximum of the mean value can be derived to be

$$\begin{aligned}
\max \langle \mathcal{B}_4^{12} \rangle &= 2^4 \max \{ \langle \vec{b}_1, \sum_{k,l=1}^3 b_3^k b_4^l T_{kl}^{12} \vec{a}_2 \rangle \cos \alpha_1 \cos \alpha_3 \cos \alpha_4 \\
&\quad + \langle \vec{b}_1, \sum_{k,l=1}^3 b_3^k b_4^l T_{kl}^{12} \vec{a}_2' \rangle \sin \alpha_1 \cos \alpha_3 \cos \alpha_4 \\
&\quad + \langle \vec{b}_3, T_{00}^{12} \vec{b}_4 \rangle \sin \alpha_3 \cos \alpha_4 \} + \langle \vec{b}_4, \vec{T}_{000}^{12} \rangle \sin \alpha_4 \\
&= 2^4 \max \{ \langle \vec{b}_1, \sum_{k,l=1}^3 b_3^k b_4^l T_{kl}^{12} \vec{a}_2 \rangle^2 + \langle \vec{b}_1, \sum_{k,l=1}^3 b_3^k b_4^l T_{kl}^{12} \vec{a}_2' \rangle^2 + \langle \vec{b}_3, T_{00}^{12} \vec{b}_4 \rangle^2 + \langle \vec{b}_4, \vec{T}_{000}^{12} \rangle^2 \}^{\frac{1}{2}} \\
&= 2^4 \max \{ \lambda_1^{12}(\vec{b}_3 \vec{b}_4) + \lambda_2^{12}(\vec{b}_3 \vec{b}_4) + \|\vec{T}_{00}^{12} \vec{b}_4\|^2 - \langle \vec{b}_3, T_{00}^{12} \vec{b}_4 \rangle^2 + \|\vec{T}_{000}^{12}\|^2 - \langle \vec{b}_4, \vec{T}_{000}^{12} \rangle^2 \}^{\frac{1}{2}},
\end{aligned}$$

where $\lambda_1^{12}(\vec{b}_3 \vec{b}_4)$ and $\lambda_2^{12}(\vec{b}_3 \vec{b}_4)$ are the two greater eigenvalues of the matrix $(M^{12})^\dagger M^{12}$, $M^{12} = \sum_{k,l=1}^3 b_3^k b_4^l T_{kl}^{12}$; T_{kl}^{12} stand for the matrices with entries $(T_{kl}^{12})_{ij} = T_{ijkl}$ with $i, j, k, l = 1, 2, 3$; T_{00}^{12} is a matrix with entries $(T_{00}^{12})_{kl} = T_{00kl}$, and \vec{T}_{000}^{12} is a vector with components $(\vec{T}_{000}^{12})_l = T_{000l}$, $l = 1, 2, 3$. The maximum in the last equation is taken over all the unit vectors \vec{b}_3 and \vec{b}_4 .

In terms of the analysis above, for four-qubit systems we have the following Theorem.

Theorem 3: The maximum of the mean values of the Bell operators in (5) for four qubits systems are given by the following formula:

$$\max |\langle \mathcal{B}_4^m \rangle| = 2^4 \max \{ \lambda_1^m(\vec{b}_3 \vec{b}_4) + \lambda_2^m(\vec{b}_3 \vec{b}_4) + \|\vec{T}_{00}^m \vec{b}_4\|^2 - \langle \vec{b}_3, T_{00}^m \vec{b}_4 \rangle^2 + \|\vec{T}_{000}^m\|^2 - \langle \vec{b}_4, \vec{T}_{000}^m \rangle^2 \}^{\frac{1}{2}}. \quad (24)$$

The maximums on the right side are taken over all the unit vectors \vec{b}_3 and \vec{b}_4 . Here $\lambda_1^m(\vec{b}_3 \vec{b}_4)$ and $\lambda_2^m(\vec{b}_3 \vec{b}_4)$ are the two greater eigenvalues of the matrix $(M^m)^\dagger M^m$, $M^m = \sum_{k,l=1}^3 b_3^k b_4^l T_{kl}^m$, $m = 1, 2, \dots, \frac{N!}{2}$; T_{kl}^m are the matrices with entries $(T_{kl}^1)_{ij} = T_{likj}$, $(T_{kl}^2)_{ij} = T_{likj}$, $(T_{kl}^3)_{ij} = T_{likj}$, $(T_{kl}^4)_{ij} = T_{likj}$, $(T_{kl}^5)_{ij} = T_{likj}$, $(T_{kl}^6)_{ij} = T_{likj}$, $(T_{kl}^7)_{ij} = T_{likj}$, $(T_{kl}^8)_{ij} = T_{likj}$, $(T_{kl}^9)_{ij} = T_{likj}$, $(T_{kl}^{10})_{ij} = T_{likj}$, $(T_{kl}^{11})_{ij} = T_{likj}$ and $(T_{kl}^{12})_{ij} = T_{likj}$, $i, j = 1, 2, 3$ and $k, l = 0, 1, 2, 3$; \vec{T}_{000}^m stand for the vectors with components $(\vec{T}_{000}^i)_x = T_{x000}$, $(\vec{T}_{000}^j)_x = T_{0x00}$, $(\vec{T}_{000}^k)_x = T_{00x0}$, $(\vec{T}_{000}^l)_x = T_{000x}$, $i = 1, 2, 3$, $j = 4, 5, 6$, $k = 7, 8, 9$, $l = 10, 11, 12$ and $x = 1, 2, 3$.

As an example, consider the 4-qubit W state $|W\rangle = \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle)$, by using the formula (24) one gets the maximal mean value $\max |\langle \mathcal{B}_4^{12} \rangle| = 1.118$. For the mixed state $\rho = \frac{x}{16}I + (1-x)|W\rangle\langle W|$, entanglement can be detected by (24) for $0 \leq x \leq 0.106$.

Generally, for any N-qubit quantum state, the maximal mean values of the Bell operators in (5) can be calculated similarly by using our approach above. For example, the maximal mean value of $\mathcal{B}_N^{\frac{N!}{2}}$ can be expressed as

$$\begin{aligned}
\max |\langle \mathcal{B}_N^{\frac{N!}{2}} \rangle| &= 2^N \max \{ \lambda_1^{\frac{N!}{2}}(\vec{b}_3 \cdots \vec{b}_N) + \lambda_2^{\frac{N!}{2}}(\vec{b}_3 \cdots \vec{b}_N) + \|\vec{T}_{45 \dots N}\|^2 - \langle \vec{b}_3, \vec{T}_{45 \dots N} \rangle^2 \\
&\quad + \|\vec{T}_{5 \dots N}\|^2 - \langle \vec{b}_4, \vec{T}_{5 \dots N} \rangle^2 + \cdots + \|\vec{T}_N\|^2 - \langle \vec{b}_N, \vec{T}_N \rangle^2 \}^{\frac{1}{2}}, \quad (25)
\end{aligned}$$

where $\lambda_1^{\frac{N!}{2}}(\vec{b}_3 \cdots \vec{b}_N)$ and $\lambda_2^{\frac{N!}{2}}(\vec{b}_3 \cdots \vec{b}_N)$ are the two greater eigenvalues of the matrix $M^\dagger M$, with $(M)_{ij} = \sum_{i_3, \dots, i_N=1}^3 b_3^{i_3} \cdots b_N^{i_N} T_{ij i_3 \dots i_N}$ the entries of matrix M ; $\vec{T}_{45 \dots N}$, $\vec{T}_{5 \dots N}$ and \vec{T}_N are vectors with

components $(\vec{T}_{45\dots N})_k = \sum_{i_4, \dots, i_N=1}^3 b_4^{i_4} \cdots b_N^{i_N} T_{00ki_4, \dots, i_N}$, $(\vec{T}_{5\dots N})_k = \sum_{i_5, \dots, i_N=1}^3 b_5^{i_5} \cdots b_N^{i_N} T_{000ki_5, \dots, i_N}$ and $(\vec{T}_N)_k = T_{000\dots 0k}$ respectively. The maximum on the right side is taken over all the unit vectors $\vec{b}_3, \vec{b}_4, \dots, \vec{b}_N$. The other mean values of the Bell operators in (5) for N -qubit states can be obtained similarly. By expressing the unit vectors \vec{b}_k as $(\cos \theta_k \cos \phi_k, \cos \theta_k \sin \phi_k, \sin \theta_k)$, $k = 3, \dots, N$, our formulas can be used to compute the maximal violation by searching for the maximum over all θ_k and ϕ_k , either analytically or numerically.

In conclusion, we have presented a series of Bell inequalities for multipartite qubits systems. These Bell inequalities are more effective in detecting the non-local properties of quantum states that can not be described by local realistic models. Formulas are derived to calculate the maximal violation of the Bell inequalities for any given multiqubit states, which gives rise to the sufficient and necessary condition for the violation of these Bell inequalities. For a fixed multiqubit state, one can optimize the mean values of the Bell operators over all measurement directions. Our Bell operators only involve two measurement settings per site, which meets the simplicity requirements of current linear optical experiments for nonlocality tests. Moreover, our formulas for the maximal violation of the Bell inequalities fit for both pure and mixed states, and can be used to improve the detection of multiqubit entanglement.

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